

# Remarks On A Nicolas Inequality

Boris A. Kupershmidt

The University of Tennessee Space Institute

Tullahoma, TN 37388, USA

E-mail: bkupersh@utsi.edu

## Abstract

The Nicolas Conjecture appears to be true.

## 1 Introduction

The Nicolas Conjecture [Nic 1983] states that

$$\frac{\mathcal{N}_k}{\varphi(\mathcal{N}_k)} > e^\gamma \log \log \mathcal{N}_k, \quad k \geq 1, \quad (1)$$

where:

$$\mathcal{N}_k = \prod_{i=1}^k p_i,$$

$p_i$  is the prime number  $\#i$ ,  $\varphi$  is the Euler phi-function, and  $\gamma = 0.57\dots$  is the Euler constant. For more details, see the beautiful paper [CLM 2006], where it was proven that

$$\prod_{i=1}^k (p_i + 1)/p_i < e^\gamma \log \log \mathcal{N}_k, \quad k > 4, \quad (2)$$

in contrast to (1):

$$\prod_{i=1}^k p_i/(p_i - 1) > e^\gamma \log \log \mathcal{N}_k, \quad k \geq 1. \quad (3)$$

Crucially, Nicolas proved that if his Conjecture is *not* true then the inequality (1) is both true and untrue infinitely often. Thus, it's enough to establish it for  $k \gg 1$ , i.e. for  $k$  large enough.

## 2 The Method

The Conjecture, *as it stands*, is rather difficult to prove with modern technical tools (see [DuS 1998], [RSh 1962, 1975]), because these tools are unable to capture the miniscule difference between the LHS and the RHS of (1).

But the numerical data reveal that the ratios

$$\text{Nic}_k = \frac{\mathcal{N}_k}{\varphi(\mathcal{N}_k)} / e^\gamma \log \log \mathcal{N}_k \quad (4)$$

approach 1 *monotonically*. If this fact could be proven, it will establish the Conjecture, since ([Lan 1909])

$$\lim_{k \rightarrow \infty} \mathcal{N}ic_k = 1. \quad (5)$$

(Mersen's estimate gives a very short proof of (5).)

Unfortunately, the modern technical tools are insufficient to prove the strengthened Nicolas Conjecture (*SNC*)

$$\mathcal{N}ic_k > \mathcal{N}ic_{k+1}. \quad (6)$$

Clearly, further strengthening is required.

Denote:

$$q = p_n, Q = \log q; p = p_{n+1}, P = \log p; \pi_n = \log \mathcal{N}_n; \quad (7)$$

$$x' = x \left(1 + \frac{1/2}{\log^2 x}\right). \quad (8)$$

We are going to use a few estimates, assembled together in Dusart's wonderful thesis. The most important estimate is:

$$p_{k+1} \leq p_k \left(1 + \frac{1/2}{\log^2 p_k}\right), \quad k \geq 463. \quad (9)$$

From now on,  $n > 463$ .

Set:

$$F_n = \varphi_n(x)|_{x=p_n}, \quad (10a)$$

$$\varphi_n(x) = \frac{x}{E(n)}, \quad E(n) = n(\log n + 2\log \log n), \quad (10b)$$

so that

$$F_n < 1, \quad \lim_{n \rightarrow \infty} F_n = 1, \quad (10c)$$

and define

$$\mathcal{N}ic'_n = \mathcal{N}ic_n F_n \Rightarrow \quad (11a)$$

$$\mathcal{N}ic_n = \mathcal{N}ic'_n / F_n > \mathcal{N}ic'_n. \quad (11b)$$

We first conjecture (*S<sup>2</sup>NC*) that

$$\mathcal{N}ic'_n > 1, \quad (12)$$

and since it's also too difficult, to have a chance to prove it we modify it into *S<sup>3</sup>NC*:

$$\mathcal{N}ic'_n > \mathcal{N}ic'_{n+1}, \quad n \gg 1. \quad (13)$$

This will be enough: if  $\mathcal{N}ic'_n$  ever dips below 1, it will stay below that level, and  $F_n \rightarrow 1$ : a contradiction to the Nicolas and Landau results; and if  $\mathcal{N}ic'_n > 1$  forever, then so will be  $\mathcal{N}ic_n > \mathcal{N}ic'_n$ .

### 3 The Proof

Writing (13) in the long-hand, we get:

$$\mathcal{N}ic'_n = F_n \mathcal{N}ic_n \stackrel{?}{>} F_{n+1} \mathcal{N}ic_n \frac{p}{p-1} \frac{\log \pi_n}{\log(\pi_n + P)} \Leftrightarrow \quad (14a)$$

$$F_{n+1} \frac{p}{p-1} \frac{\log \pi_n}{\log(\pi_n + P)} \stackrel{?}{<} F_n. \quad (14b)$$

Consider the function  $z \mapsto \frac{\log z}{\log(z+a)}$ . It's monotonically increasing with  $z$  when  $a > 0$ , so that the LHS of (14b) will *increase* when  $\pi_n$  reaches its maximal range. Since

$$\pi_n \leq q \left(1 + \frac{c_s}{Q^s}\right), s = 1, 2, 3, \dots, \quad (15a)$$

and we are going to calculate mod  $\frac{1}{q^2}$  and mod  $\frac{1}{q} \frac{1}{Q^3}$ , we take  $s = 3$ . (It will be seen from the Proof that this simplification makes no difference in the end.) We treat  $q$  as a variable. Thus,

$$\pi = \pi_n|_{max} = q \Rightarrow \quad (15b)$$

$$\frac{\log \pi_n}{\log(\pi_n + P)} = \frac{\log \pi_n}{\log \pi_n + \frac{P}{\pi_n}} = 1 - \frac{P}{qQ}. \quad (16)$$

Next, the LHS of (14b) is the value at  $x = p$  of the function

$$f(x) := \frac{x}{E(n+1)} \left(1 + \frac{1}{x-1}\right) \left(1 - \frac{\log x}{qQ}\right). \quad (17)$$

**Lemma 18.** The function  $f(x)$  is increasing with  $x$  on the interval

$$q \leq x \leq q' = q \left(1 + \frac{1/2}{Q^2}\right) \quad (19)$$

**Proof.** Note that  $p_{n+1} \geq p_n + 2$ , but we are generously allowing  $x$  to creep to  $q$  instead of  $q + 2$ .

Calculating  $\frac{\partial f}{\partial x}$ , we find:

$$1 - \frac{x}{qQ} \stackrel{?}{>} (x+1) \frac{1}{xqQ} \Leftrightarrow \quad (20a)$$

$$1 \stackrel{?}{>} \frac{x}{qQ} + \frac{1}{qQ} + \frac{1}{xqQ}, \quad (20b)$$

which is obvious ■

Thus,  $f(x)$  is maximized when  $x$  is, and

$$x \leq x_{max} = q' = q \left(1 + \frac{1/2}{Q^2}\right). \quad (21)$$

Set  $F_{n+1}$  accordingly:

$$\overline{F}_{n+1} = \varphi_n(x)|_{x=q'} = \frac{q'}{E(n+1)}. \quad (22)$$

Our inequality (14b) turns into

$$\left(1 + \frac{1}{q'}\right) \left(1 - \frac{Q + \frac{1/2}{Q^2}}{qQ}\right) \stackrel{?}{<} \frac{F_n}{\overline{F}_{n+1}} \Leftrightarrow \quad (23a)$$

$$\left[1 + \frac{1}{q(1 + \frac{1/2}{Q^2})}\right] \left(1 - \frac{1 + \frac{1/2}{Q^3}}{q}\right) \stackrel{?}{<} \frac{E(n+1)}{E(n)} \frac{1}{1 + \frac{1/2}{Q^2}} = \frac{E(n+1)}{E(n)} \left(1 - \frac{1/2}{Q^2}\right). \quad (23b)$$

Now, for the LHS of (23b) we find:

$$\left[1 + \frac{1}{q} \left(1 - \frac{1/2}{Q^2}\right)\right] \left(1 - \frac{1}{q}\right) = 1 - \frac{1/2}{qQ^2} < 1, \quad (24)$$

while for the RHS of (23b) we find, first:

$$\begin{aligned} \frac{E(n+1)}{E(n)} &= \frac{n(1 + \frac{1}{n})[\log n + \frac{1}{2} + 2\log\log n + \frac{2}{n\log\log n}]}{n(\log n + 2\log\log n)} \quad [\text{neglecting smaller terms}] \\ &= \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n\log n}\right) > 1 + \frac{1}{n}. \end{aligned} \quad (25)$$

Since

$$\frac{1}{n} \leq \frac{Q}{q}, \quad n \geq 4, \quad (26a)$$

$$\frac{1}{n} \geq \frac{Q-2}{q}, \quad n \geq 5, \quad (26b)$$

we have:

$$\begin{aligned} \frac{1}{n} &= \frac{Q - a_n}{q}, \quad 0 \leq a_n \leq 2, \quad n \geq 5 \Rightarrow \\ \frac{1}{n} &\sim \frac{Q}{q}, \end{aligned} \quad (27)$$

and the RHS of (23b) returns:

$$1 + \frac{Q}{q} \left(1 - \frac{1/2}{Q^2}\right) = 1 + \frac{Q}{q} + \dots > 1 > LHS = 1 - \frac{1/2}{qQ^2} \blacksquare \blacksquare \quad (28)$$

## 4 Acknowledgement 1

Although not apparent, the paper is a result of hundreds of computer experiments, and it has an invisible hero: the PARI-GP program, a free to the mathematical community wonderful tool for computer experiments. My deep gratitude to the developers of PARI.

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