

Remarks On A Nicolas Inequality

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Abstract

The Nicolas Conjecture appears to be true.

1 Introduction

The Nicolas Conjecture [Nic 1983] states that

$$\frac{\mathcal{N}_k}{\varphi(\mathcal{N}_k)} > e^\gamma \log \log \mathcal{N}_k, \quad k \geq 1, \quad (1)$$

where:

$$\mathcal{N}_k = \prod_{i=1}^k p_i,$$

p_i is the prime number $\#i$, φ is the Euler phi-function, and $\gamma = 0.57\dots$ is the Euler constant. For more details, see the beautiful paper [CLM 2006], where it was proven that

$$\prod_{i=1}^k (p_i + 1)/p_i < e^\gamma \log \log \mathcal{N}_k, \quad k > 4, \quad (2)$$

in contrast to (1):

$$\prod_{i=1}^k p_i/(p_i - 1) > e^\gamma \log \log \mathcal{N}_k, \quad k \geq 1. \quad (3)$$

Crucially, Nicolas proved that if his Conjecture is *not* true then the inequality (1) is both true and untrue infinitely often. Thus, it's enough to establish it for $k \gg 1$, i.e. for k large enough.

2 The Method

The Conjecture, *as it stands*, is rather difficult to prove with modern technical tools (see [DuS 1998], [RSh 1962, 1975]), because these tools are unable to capture the miniscule difference between the LHS and the RHS of (1).

But the numerical data reveal that the ratios

$$\text{Nic}_k = \frac{\mathcal{N}_k}{\varphi(\mathcal{N}_k)} / e^\gamma \log \log \mathcal{N}_k \quad (4)$$

approach 1 *monotonically*. If this fact could be proven, it will establish the Conjecture, since ([Lan 1909])

$$\lim_{k \rightarrow \infty} \mathcal{N}ic_k = 1. \quad (5)$$

(Mersen's estimate gives a very short proof of (5).)

Unfortunately, the modern technical tools are insufficient to prove the strengthened Nicolas Conjecture (*SNC*)

$$\mathcal{N}ic_k > \mathcal{N}ic_{k+1}. \quad (6)$$

Clearly, further strengthening is required.

Denote:

$$q = p_n, Q = \log q; p = p_{n+1}, P = \log p; \pi_n = \log \mathcal{N}_n; \quad (7)$$

$$x' = x \left(1 + \frac{1/2}{\log^2 x}\right). \quad (8)$$

We are going to use a few estimates, assembled together in Dusart's wonderful thesis. The most important estimate is:

$$p_{k+1} \leq p_k \left(1 + \frac{1/2}{\log^2 p_k}\right), \quad k \geq 463. \quad (9)$$

From now on, $n > 463$.

Set:

$$F_n = \varphi_n(x)|_{x=p_n}, \quad (10a)$$

$$\varphi_n(x) = \frac{x}{E(n)}, \quad E(n) = n(\log n + 2\log \log n), \quad (10b)$$

so that

$$F_n < 1, \quad \lim_{n \rightarrow \infty} F_n = 1, \quad (10c)$$

and define

$$\mathcal{N}ic'_n = \mathcal{N}ic_n F_n \Rightarrow \quad (11a)$$

$$\mathcal{N}ic_n = \mathcal{N}ic'_n / F_n > \mathcal{N}ic'_n. \quad (11b)$$

We first conjecture (*S²NC*) that

$$\mathcal{N}ic'_n > 1, \quad (12)$$

and since it's also too difficult, to have a chance to prove it we modify it into *S³NC*:

$$\mathcal{N}ic'_n > \mathcal{N}ic'_{n+1}, \quad n \gg 1. \quad (13)$$

This will be enough: if $\mathcal{N}ic'_n$ ever dips below 1, it will stay below that level, and $F_n \rightarrow 1$: a contradiction to the Nicolas and Landau results; and if $\mathcal{N}ic'_n > 1$ forever, then so will be $\mathcal{N}ic_n > \mathcal{N}ic'_n$.

3 The Proof

Writing (13) in the long-hand, we get:

$$\mathcal{N}ic'_n = F_n \mathcal{N}ic_n \stackrel{?}{>} F_{n+1} \mathcal{N}ic_n \frac{p}{p-1} \frac{\log \pi_n}{\log(\pi_n + P)} \Leftrightarrow \quad (14a)$$

$$F_{n+1} \frac{p}{p-1} \frac{\log \pi_n}{\log(\pi_n + P)} \stackrel{?}{<} F_n. \quad (14b)$$

Consider the function $z \mapsto \frac{\log z}{\log(z+a)}$. It's monotonically increasing with z when $a > 0$, so that the LHS of (14b) will *increase* when π_n reaches its maximal range. Since

$$\pi_n \leq q \left(1 + \frac{c_s}{Q^s}\right), s = 1, 2, 3, \dots, \quad (15a)$$

and we are going to calculate mod $\frac{1}{q^2}$ and mod $\frac{1}{q} \frac{1}{Q^3}$, we take $s = 3$. (It will be seen from the Proof that this simplification makes no difference in the end.) We treat q as a variable. Thus,

$$\pi = \pi_n|_{max} = q \Rightarrow \quad (15b)$$

$$\frac{\log \pi_n}{\log(\pi_n + P)} = \frac{\log \pi_n}{\log \pi_n + \frac{P}{\pi_n}} = 1 - \frac{P}{qQ}. \quad (16)$$

Next, the LHS of (14b) is the value at $x = p$ of the function

$$f(x) := \frac{x}{E(n+1)} \left(1 + \frac{1}{x-1}\right) \left(1 - \frac{\log x}{qQ}\right). \quad (17)$$

Lemma 18. The function $f(x)$ is increasing with x on the interval

$$q \leq x \leq q' = q \left(1 + \frac{1/2}{Q^2}\right) \quad (19)$$

Proof. Note that $p_{n+1} \geq p_n + 2$, but we are generously allowing x to creep to q instead of $q + 2$.

Calculating $\frac{\partial f}{\partial x}$, we find:

$$1 - \frac{x}{qQ} \stackrel{?}{>} (x+1) \frac{1}{xqQ} \Leftrightarrow \quad (20a)$$

$$1 \stackrel{?}{>} \frac{x}{qQ} + \frac{1}{qQ} + \frac{1}{xqQ}, \quad (20b)$$

which is obvious ■

Thus, $f(x)$ is maximized when x is, and

$$x \leq x_{max} = q' = q \left(1 + \frac{1/2}{Q^2}\right). \quad (21)$$

Set F_{n+1} accordingly:

$$\overline{F}_{n+1} = \varphi_n(x)|_{x=q'} = \frac{q'}{E(n+1)}. \quad (22)$$

Our inequality (14b) turns into

$$\left(1 + \frac{1}{q'}\right) \left(1 - \frac{Q + \frac{1/2}{Q^2}}{qQ}\right) \stackrel{?}{<} \frac{F_n}{\overline{F}_{n+1}} \Leftrightarrow \quad (23a)$$

$$\left[1 + \frac{1}{q(1 + \frac{1/2}{Q^2})}\right] \left(1 - \frac{1 + \frac{1/2}{Q^3}}{q}\right) \stackrel{?}{<} \frac{E(n+1)}{E(n)} \frac{1}{1 + \frac{1/2}{Q^2}} = \frac{E(n+1)}{E(n)} \left(1 - \frac{1/2}{Q^2}\right). \quad (23b)$$

Now, for the LHS of (23b) we find:

$$\left[1 + \frac{1}{q} \left(1 - \frac{1/2}{Q^2}\right)\right] \left(1 - \frac{1}{q}\right) = 1 - \frac{1/2}{qQ^2} < 1, \quad (24)$$

while for the RHS of (23b) we find, first:

$$\begin{aligned} \frac{E(n+1)}{E(n)} &= \frac{n(1 + \frac{1}{n})[\log n + \frac{1}{2} + 2\log\log n + \frac{2}{n\log\log n}]}{n(\log n + 2\log\log n)} \quad [\text{neglecting smaller terms}] \\ &= \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{n\log n}\right) > 1 + \frac{1}{n}. \end{aligned} \quad (25)$$

Since

$$\frac{1}{n} \leq \frac{Q}{q}, \quad n \geq 4, \quad (26a)$$

$$\frac{1}{n} \geq \frac{Q-2}{q}, \quad n \geq 5, \quad (26b)$$

we have:

$$\begin{aligned} \frac{1}{n} &= \frac{Q - a_n}{q}, \quad 0 \leq a_n \leq 2, \quad n \geq 5 \Rightarrow \\ \frac{1}{n} &\sim \frac{Q}{q}, \end{aligned} \quad (27)$$

and the RHS of (23b) returns:

$$1 + \frac{Q}{q} \left(1 - \frac{1/2}{Q^2}\right) = 1 + \frac{Q}{q} + \dots > 1 > LHS = 1 - \frac{1/2}{qQ^2} \blacksquare \blacksquare \quad (28)$$

4 Acknowledgement 1

Although not apparent, the paper is a result of hundreds of computer experiments, and it has an invisible hero: the PARI-GP program, a free to the mathematical community wonderful tool for computer experiments. My deep gratitude to the developers of PARI.

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5 Acknowledgement 2

This paper might have never been finished were it not for the vaguely unthreatening letter from Thi-Eve Jao, the Editor-in-Chief of the International Journal of Chinese Mathematics, in which he politely explained that idleness of people like me threatens the livelihood of people like him, and implored me to finish the proof of the Nicolas conjecture as expeditiously as possible, but no later than the opening of the next Congress of the Chinese Communist Gangsters Party. There was nothing else to do but to comply with so exquisitely framed request.

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