NONCOMMUTATIVE INTEGRABLE SYSTEMS OF HYDRODYNAMIC TYPE

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Abstract

On noncommutative spaces, integrable hierarchies of hydrodynamic type systems (1^{st}-order quasi-linear PDE's) do not, in general, exist. An infinite-component exception to this general observation is analysed in detail.

1 Introduction

Systems of hydrodynamic type, in one space dimension, have the form

$$\frac{\partial u_i}{\partial t} = \sum_j F_i^j(\boldsymbol{u}) \frac{\partial u_j}{\partial x}. \tag{1.1}$$

The Euler equations of ideal fluids and plasmas are of this form (in any dimension; see, e.g. [7]); the Benney equations [1]

$$A_{n,t} = A_{n+1,x} + nA_{n-1}A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.2}$$

which are the quasiclassical limit [10] of the 2^{nd} flow in the KP hierarchy, are of this form; and the list goes on and on (see, e.g., [3, 4, 11, 12, 13].) In short, these are ubiquitous systems.

Viewed as zero-dispersion limits of more general dispersive dynamical systems, they are clearly a purely *commutative* phenomenon: in the noncommutative realm, there are almost always non-derivative terms present (such as commutators) which preclude the passage to the zero-dispersion limit. The typical example is the DWW (Dispersive Water Waves) system from [9, Ch. 2,3]:

$$a_{0,t} = -\frac{1}{2}a_0'' + a_0' a_0 + a_1' + [a_0, a_1], \tag{1.3a}$$

$$a_{1,t} = \frac{1}{2}a_1'' + (a_0 a_1)', \tag{1.3b}$$

where $(\cdot)' = (\cdot)_{,x}$ stands for the x-derivative of $(\cdot)$. *Only* in the commutative case, when the commutator $[a_0, a_1]$ in the equation $(1.3a)$ drops out, can one pass to the zero-dispersion limit and recover the classical long wave system

$$u_t = uu_x + h_x, \tag{1.4a}$$

$$h_t = (uh)_x, \tag{1.4b}$$

with $u = a_0,\ h = a_1$. The hydrodynamical type system (1.4) doesn't have a noncommutative analog which remains of hydrodynamic type. The same conclusion applies to the Benney system: the full noncommutative KP flow #2 has the form [9, Ch. 1]

$$A_{n,t} = \frac{1}{2}A_n^{(2)} + A_{n+1}^{(1)} - \sum_{i+k=n} (-1)^k \binom{n}{k} A_i A_0^{(k)} + A_0 A_n, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.5}$$

with $(\cdot)^{(k)}$ standing for $\partial^k(\cdot)$, $\ \partial = \partial/\partial x$.

In general, one shouldn't expect hydrodynamical systems to exist in the noncommutative world. Nevertheless, there are at least two exceptions known.

The 1^{st} one is the Svinolupov KdV hierarchy with the values in a Jordan algebra [16, 17], with the first nontrivial flow of the form

$$u_t = u_{xxx} + 6u \circ u_x; \tag{1.6}$$

here u takes values in an arbitrary fixed Jordan algebra $\mathcal{A}$ with multiplication $\circ$. The zero-dispersion limit of the equation (1.6),

$$u_t = 6u \circ u_x \tag{1.7}$$

is of hydrodynamic type, and the whole hierarchy is as well (see [14, 15]). When the Jordan algebra $\mathcal{A}$ comes out of an associative algebra, the equation (1.7) reduces to

$$u_t = const(u^2)_x, \tag{1.8}$$

and all the higher flows have the form

$$u_t = const_n(u^n)_x, \quad n \in \mathbf{Z}_{\geq 1}. \tag{1.9}$$

The 2^{nd} exceptional system is the noncommutative Burgers hierarchy with the values in an arbitrary associative algebra [8; 9, §2.5], with the first nontrivial flow of the form

$$u_t = u_{xx} + 2uu_x. \tag{1.10}$$

The zero-dispersion limit of this equation is

$$u_t = const\,uu_x, \quad const = 2, \tag{1.11}$$

and the whole commuting hierarchy has the form

$$u_t = const_n u^n u_x. \tag{1.12}$$

Both of the exceptional hydrodynamic hierarchies discussed above operate with a *single scalar* field, u. One might tentatively conclude that scalarity to be essential for the existence of noncommutative integrable hydrodynamic hierarchies. Fortunately, the scalarity is a historical accident.

Consider the infinite-component system

$$A_{n,t} = A_{n+1,x} + bA_n A_{0,x}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.13}$$

where b is a fixed constant commuting with everything, and all the variables are noncommutative but associative, like in [9]. It is shown below that the system (1.13) is *integrable*: it has an infinite number of mutually commuting higher flows (§2), all of them having a common infinite set of conserved densities (§3), and all of them of hydrodynamical type (§2). This hierarchy, in the general commutative form

$$\dot{A}_n = A_{n+1,x} + (an+b)A_n A_{0,x} + cA_0 A_{n,x}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.14}$$

was introduced in [6]. It is the lift into the moments space

$$A_n(x,t) = \int_0^h u^n(x,y,t)dy, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.15}$$

of the 2-dimensional free-surface hydrodynamical system

$$u_t = uu_x + auh_x + chu_x - u_y \int_0^y dy(u_x + bh_x), \tag{1.16a}$$

$$h_t = (\int_0^h udy + \frac{b+c}{2}h^2)_x, \tag{1.16b}$$

where $u = u(x,y,t)$ and $h = h(x,t)$. When u is y-independent, $u_y = 0$, the system (1.16) becomes (for the case of interest, $a = c = 0$):

$$u_t = uu_x, \tag{1.17a}$$

$$h_t = (uh + \frac{b}{2}h^2)_x. \tag{1.17b}$$

Therefore, the map (1.15),

$$A_n = hu^n, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.18}$$

maps the 2-component system (1.17) into the infinite-component system (1.19) (with $a = c = 0$).

No free-surface interpretation of the type (1.16) is available in the noncommutative case; nevertheless, the reduction (1.18) not only survives but, in contrast to the framework of Lax representations (see [9, Ch. 14]), has two *different* versions:

$$\begin{aligned} A_n &= u^n h, \\ u_t &= u_x u, \\ h_t &= (uh)_x + hh_x, \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} A_n &= hu^n, \\ u_t &= uu_x + [u, h_x], \\ h_t &= (hu)_x + hh_x, \end{aligned} \tag{1.20}$$

with the constant b being scaled away into $b = 1$. These reduction formulae, generalized to the whole commuting hierarchy, are proved in §4. Interpreting these reductions leads, in §5 and §6, into the construction of two modified hierarchies. A single-component reduction, $A_k = (-1)^k A_0^{k+1}$, is proven in §7 to work for the whole hierarchy.

Studying generalized symmetries of the heat equation, Bluman and Cole arrived at the following system [2, p. 1041]:

$$-\frac{1}{2}A_t = C_x + AA_x - \frac{1}{2}A_{xx}, \tag{1.21a}$$

$$-\frac{1}{2}C_t = CA_x - \frac{1}{2}C_{xx}. \tag{1.21b}$$

Subsequently, Harrison and Estabrook [5] gave another derivation of the system (1.21); more recently, this system was studied by Webb [18]. The system (1.21) doesn't appear to be integrable, although a proof of nonintegrability is still lacking. However, the quasiclassical limit of this system is certainly integrable, for (with $-\frac{1}{2}\partial_t$ replaced by ∂_t) it is nothing but our system (1.13) in its commutative version, with $b = 1$, with all the variables A_n for $n > 1$ set to zero; and with the remaining variables A_0 and A_1 renamed by A and C, respectively.

2 Construction Of Higher Flows

The rescaling

$$A_n \mapsto b^n A_n, \quad \partial_t \mapsto b\partial_t, \tag{2.1}$$

makes $b \neq 0$ into $b = 1$ in the system (1.13):

$$X_2(A_n) = \dot{A}_n = A'_{n+1} + A_n A'_0, \quad n \in \mathbf{Z}_{\geq 0}. \tag{2.2}$$

This is the 2^{nd} flow of our hierarchy, the 1^{st} one being

$$X_1(A_n) = A'_n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{2.3}$$

Thus

$$\boldsymbol{X}_2 = \mathcal{M}\boldsymbol{X}_1, \tag{2.4}$$

where, for any evolutionary derivation $X = \hat{X}$,

$$\boldsymbol{X} = X(\boldsymbol{A}), \tag{2.5}$$

and

$$\mathcal{M}_i^j = \delta_{i+1}^j + \delta_0^j v_i, \quad i, j \in \mathbf{Z}_{\geq 0}, \tag{2.6}$$

$$v_i = \hat{L}_{A_i}; \tag{2.7}$$

where $\hat{L}_f$ and $\hat{R}_f$ denote the operators of left and right multiplication:

$$fg = \hat{L}_f(g) = \hat{R}_g(f). \tag{2.8}$$

The higher flows are constructed by the formula

$$\boldsymbol{X}_{L+1} = \mathcal{M}^L(\boldsymbol{X}_1), \quad L \in \mathbf{Z}_{\geq 0}. \tag{2.9}$$

To make it more explicit, we consider the matrix $\mathcal{M}$ (2.6) where the v_i's are considered as arbitrary free noncommuting but associative variables.

Proposition 2.10. For $L \in \mathbf{Z}_{\geq 1}$,

$$(\mathcal{M}^L)_i^j = \delta_{i+L}^j + \sum_{s=0}^{L-1} \delta_s^j \sum_{k=0}^{L-1-s} v_{i+L-1-s-k}\mathcal{E}_k = \tag{2.11a}$$

$$= \delta_{i+L}^j + \sum_{s=0}^{L-1} \delta_{L-1-s}^j \sum_{r=0}^{s} v_{i+s-r}\mathcal{E}_r, \tag{2.11b}$$

where the polynomials $\mathcal{E}_i$'s are defined by the rule:

$$\mathcal{E}_0 = 1; \quad \mathcal{E}_L = \sum_{s=0}^{L-1} \mathcal{E}_s v_{L-1-s}, \quad L \in \mathbf{Z}_{\geq 1}. \tag{2.12}$$

Proof. Formulae (2.11) are true for $L = 1$. Induction on L gives:

$$(\mathcal{M}^{L+1})_i^j = \sum_k (\mathcal{M}^L)_i^k \mathcal{M}_k^j =$$

$$= \sum_k \{\delta_{i+L}^k + \sum_{0\leq r\leq s\leq L-1} \delta_{L-1-s}^k v_{i+s-r}\mathcal{E}_r\}\{\delta_{k+1}^j + \delta_0^j v_k\} =$$

$$= \delta_{i+L+1}^j + \sum_{0\leq r\leq s\leq L-1} \delta_{L-s}^j v_{i+s-r}\mathcal{E}_r + \tag{2.13a}$$

$$+ \delta_0^j\{v_{i+L} + \sum_{0\leq r\leq s\leq L-1} v_{i-s-r}\mathcal{E}_r v_{L-1-s}\}. \tag{2.13b}$$

On the other hand, formula (2.11b) predicts that

$$(\mathcal{M}^{L+1})_i^j = \delta_{i+L+1}^j + \sum_{s=0}^{L} \delta_{L-s}^j \sum_{k=0}^{s} v_{i+s-k}\mathcal{E}_k. \tag{2.14}$$

Comparing formulae (2.13) and (2.14), we see that they agree provided their δ_0^j-terms do:

$$v_{i+L} + \sum_{0\leq r\leq s\leq L-1} v_{i+s-r}\mathcal{E}_r v_{L-1-s} \overset{?}{=} \sum_{k=0}^{L} v_{i+L-k}\mathcal{E}_k. \tag{2.15}$$

This is so, provided

$$\sum_{s-r=k} \mathcal{E}_r v_{L-1-s} \stackrel{?}{=} \mathcal{E}_{L-k}, \tag{2.16}$$

which is formula (2.12) ■

We next show that all the higher flows commute with the flow X_2 (2.2). Since the matrix $\mathcal{M}$ (2.6,7) acts as a genuine recursion operator, we need to check that

$$X_2(\mathcal{M}) = [D(\boldsymbol{X}_2), \mathcal{M}], \tag{2.17}$$

where $D(\cdot)$ is the Fréchet derivative of $(\cdot)$.

From formula (2.2),

$$D(\boldsymbol{X}_2)^j_i = \delta^j_{i+1}\partial + \delta^j_i \hat{R}_{A'_0} + \delta^j_0 \hat{L}_{A_i}\partial. \tag{2.18}$$

Therefore

$$\begin{aligned}
&[D(\boldsymbol{X}_2), \mathcal{M}]^j_i = \sum_k \{D(\boldsymbol{X}_2)^k_i \mathcal{M}^j_k - \mathcal{M}^k_i D(\boldsymbol{X}_2)^j_k\} = \\
&= \sum_k \{\delta^k_{i+1}\partial + \delta^k_i \hat{R}_{A'_0} + \delta^k_0 \hat{L}_{A_i}\partial\}\{\delta^j_{k+1} + \delta^j_0 \hat{L}_{A_k}\} - \\
&- \sum_k \{\delta^k_{i+1} + \delta^k_0 \hat{L}_{A_i}\}\{\delta^j_{k+1}\partial + \delta^j_k \hat{R}_{A'_0} + \delta^j_0 \hat{L}_{A_k}\partial\} = \\
&= \delta^j_{i+2}\partial + \delta^j_{i+1}\hat{R}_{A'_0} + \delta^j_1 \hat{L}_{A_i}\partial + \delta^j_0(\partial \hat{L}_{A_{i+1}} + \hat{R}_{A'_0}\hat{L}_{A_i} + \hat{L}_{A_i}\partial \hat{L}_{A_0}) - \\
&- \{\delta^j_{i+2}\partial + \delta^j_{i+1}\hat{R}_{A'_0} + \delta^j_0 \hat{L}_{A_{i+1}}\partial + \hat{L}_{A_i}(\delta^j_1\partial + \delta^j_0 \hat{R}_{A'_0} + \delta^j_0 \hat{L}_{A_0}\partial)\} = \\
&= \delta^j_0(\hat{L}_{A'_{i+1}} + \hat{L}_{A_i}\hat{L}_{A'_0}) = \delta^j_0 \hat{L}_{A'_{i+1} + A_i A'_0} = \delta^j_0 \hat{L}_{X_2(A_i)} = \\
&= X_2(\delta^j_0 \hat{L}_{A_i}) = X_2(\mathcal{M}^j_i) = X_2(\mathcal{M})^j_i.
\end{aligned} \tag{2.19}$$

Remark 2.20 Applying the anti-involution which acts identically on the A_n's, we could as well work with all our formulae mirror-inverted. Thus, the flow X_2 (2.2) would be of the form

$$X_2(A_n) = A'_{n+1} + A'_0 A_n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{2.21}$$

What would happen if we consider a *mixture* of both forms, such as

$$X_2(A_n) = A'_{n+1} + \alpha A_n A'_0 + \beta A'_0 A_n? \tag{2.22}$$

As will be seen in §§3,4, this mixture still has an infinite number of conserved densities and two hydrodynamical reductions. Nevertheless, the calculation similar to (2.19) shows that the higher flows do not commute with X_2 unless $\alpha\beta = 0$.

We next show that all the higher flows commute between themselves. The usual commutative arguments from the theory of recursion operators show that this conclusion (for any recursion *operator* $\mathcal{M}$) amounts to the identity

$$\widehat{\mathcal{M}\boldsymbol{Z}}(\mathcal{M}\boldsymbol{Y}) - \widehat{\mathcal{M}\boldsymbol{Y}}(\mathcal{M}\boldsymbol{Z}) = \mathcal{M}(\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z}), \tag{2.23}$$

for any pair of A-independent evolutionary vector fields $\hat{Y}$ and $\hat{Z}$. Now, for the case at hand,

$$\hat{Z}(\mathcal{M})^j_i = \delta^j_0 \hat{L}_{Z_i} \Rightarrow \tag{2.24a}$$

$$(\hat{Z}(\mathcal{M})\boldsymbol{Y})_k = \sum_j \hat{Z}(\mathcal{M})^j_k Y_j = Z_k Y_0 \Rightarrow \tag{2.24b}$$

$$(\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z})_k = Z_k Y_0 - Y_k Z_0 \Rightarrow \tag{2.24c}$$

$$\{\mathcal{M}(\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z})\}_i = \sum_k \mathcal{M}^k_i (Z_k Y_0 - Y_k Z_0) =$$

$$= (Z_{i+1}Y_0 - Y_{i+1}Z_0) + A_i(Z_0 Y_0 - Y_0 Z_0). \tag{2.24e}$$

On the other hand,

$$(\mathcal{M}\boldsymbol{Z})_s = \sum_\ell \mathcal{M}^\ell_s Z_\ell = Z_{s+1} + A_s Z_0 \Rightarrow \tag{2.25a}$$

$$(\widehat{\mathcal{M}\boldsymbol{Z}}(\mathcal{M}\boldsymbol{Y}))_i = \widehat{\mathcal{M}\boldsymbol{Z}}(Y_{i+1} + A_i Y_0) = \widehat{\mathcal{M}\boldsymbol{Z}}(A_i)Y_0 =$$

$$= (\mathcal{M}\boldsymbol{Z})_i Y_0 = (Z_{i+1} + A_i Z_0)Y_0 \Rightarrow \tag{2.25b}$$

$$\{\widehat{\mathcal{M}\boldsymbol{Z}}(\mathcal{M}\boldsymbol{Y}) - \widehat{\mathcal{M}\boldsymbol{Y}}(\mathcal{M}\boldsymbol{Z})\}_i = (Z_{i+1} + A_i Z_0)Y_0 - (Y_{i+1} + A_i Y_0)Z_0 =$$

$$= (Z_{i+1}Y_0 - Y_{i+1}Z_0) + A_i(Z_0 Y_0 - Y_0 Z_0), \tag{2.25c}$$

and this is the same expression as (2.24e). Thus, formula (2.23) is verified.

3 Constructions Of Integrals Of Motion

By formulae (2.9,11,7),

$$X_{L+1}(A_i) = A'_{i+L} + \sum_{s=0}^{L-1} \sum_{k=0}^{L-1-s} A_{i+L-1-s-k} \mathcal{E}_k A'_s, \quad i \in \mathbf{Z}_{\geq 0}. \tag{3.1}$$

From now on, we treat the variables v_i's in formula (2.7) not as $\hat{L}_{A_i}$'s but simply as A_i's, since the $\mathcal{E}_k$'s always act to the right (see formula (2.12).)

Set

$$\mathcal{A}(z) = \sum_{i=0}^{\infty} z^{i+1} A_i, \tag{3.2}$$

$$f(z) = 1 - \mathcal{A}(z), \tag{3.3}$$

$$\mathcal{E}(z) = \sum_{i=0}^{\infty} z^i \mathcal{E}_i, \tag{3.4}$$

$$C(z) = \sum_{k,s=0}^{\infty} z^{k+s} \mathcal{E}_k A'_s = \sum_{N=0}^{\infty} z^N C_N. \tag{3.5}$$

Proposition 3.6.

$$z^L X_{L+1}(f) = f' + f \sum_{j=0}^{L-1} z^{j+1} C_j. \tag{3.7}$$

Proof. We have, by formulae (3.1-3):

$$z^L X_{L+1}(f) = -z^L \sum_{i=0}^{\infty} z^{i+1} X_{L+1}(A_i) = -\sum_{i=0}^{\infty} z^{i+L+1} A'_{i+L} - \tag{3.8a}$$

$$-\sum_{s=0}^{L-1} \sum_{k=0}^{L-1-s} \sum_{i=0}^{\infty} z^{i+L+1} A_{i+L-1-s-k} \mathcal{E}_k A'_s. \tag{3.8b}$$

The part $(3.8a)$ is:

$$-\sum_{j\geq L} z^{j+1} A'_j = f' + \sum_{j=0}^{L-1} z^{j+1} A'_j. \tag{3.9}$$

The part $(3.8b)$ can be transformed as:

$$-\sum_{s+k\leq L-1} z^{s+k+1} (\sum_{j\geq 0} z^{j+1} A_j - \sum_{j=0}^{L-2-s-k} z^{j+1} A_j) \mathcal{E}_k A'_s =$$

$$= -\sum_{s+k\leq L-1} z^{s+k+1} (-f + 1 - \sum_{j=0}^{L-2-s-k} z^{j+1} A_j) \mathcal{E}_k A'_s =$$

$$= f \sum_{s+k\leq L-1} z^{s+k+1} \mathcal{E}_k A'_s - \sum_{s+k\leq L-1} z^{s+k+1} (1 - \sum_{j=0}^{L-2-s-k} z^{j+1} A_j) \mathcal{E}_k A'_s. \tag{3.10}$$

The 1^{st} summands in the expressions (3.9) and (3.10) combine into the RHS of formula (3.7). To show that the 2^{nd} summands cancel out, we need to verify that

$$z^{s+1} \stackrel{?}{=} \sum_{k=0}^{L-1-s} z^{s+k+1} (1 - \sum_{j=0}^{L-2-s-k} z^{j+1} A_j) \mathcal{E}_k. \tag{3.11}$$

This can be rewritten as

$$1 \stackrel{?}{=} \sum_{k=0}^{r} z^k (1 - \sum_{j=0}^{r-1-k} z^{j+1} A_j) \mathcal{E}_k = \sum_{k=0}^{r} z^k \mathcal{E}_k - z \sum_{k=0}^{r} \sum_{j=0}^{r-1-k} z^{j+r} A_j \mathcal{E}_r, \tag{3.12}$$

which amounts to

$$\mathcal{E}_0 = 1; \quad \mathcal{E}_{L+1} = \sum_{i+j=L} A_i \mathcal{E}_j. \tag{3.13}$$

This is not our definition (2.12), but it is equivalent to it, since both amount to the same identity:

$$\mathcal{E}(z) = (1 - z\mathcal{V}(z))^{-1}, \quad \mathcal{V}(z) = \sum_{i=0}^{\infty} z^i v_i, \tag{3.14}$$

one written as

$$\mathcal{E}(z) - z\mathcal{V}(z)\mathcal{E}(z) = 1, \tag{3.15a}$$

and another written as

$$\mathcal{E}(z) - z\mathcal{E}(z)\mathcal{V}(z) = 1. \quad \blacksquare \tag{3.15b}$$

Thus,

$$f^{-1}X_{L+1}(f) = z^{-L}f^{-1}f' + \sum_{j=0}^{L-1} z^{j+1-L}C_j. \tag{3.16}$$

Let us write

$$G \overset{[,]}{=} 0 \tag{3.17}$$

when G belongs to the additive subspace of commutators.
Proposition 3.18. For any derivation X,

$$X(\ell n f) \overset{[,]}{=} f^{-1}X(f). \tag{3.19}$$

Proof. This formula applies to any f of the form

$$f = 1 + F(z), \quad F(z) = \mathrm{O}(z), \tag{3.20}$$

so that the series

$$\ell n f = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} F^n \tag{3.21}$$

is well defined. Now,

$$X(\ell n f) = X(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} F^n) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{i+j=n-1} F^i X(F) F^j \overset{[,]}{=}$$
$$\overset{[,]}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{i+j=n-1} F^j F^i X(F) = \sum_{n=1}^{\infty} (-1)^{n-1} F^{n-1} X(F) = (1+F)^{-1}X(F) =$$
$$= f^{-1}X(f) \quad \blacksquare \tag{3.22}$$

Let us write

$$G \approx 0 \tag{3.23}$$

when G is "trivial" in the noncommutative sense, that is, G belongs to the additive

subspace of $Im(\partial)$ and commutators (see [9].)
Proposition 3.24.

$$C_N \approx 0, \quad N \in \mathbf{Z}_{\geq 0}. \tag{3.25}$$

Proof. We show that

$$C(z) \approx 0. \tag{3.26}$$

By formulae (3.5,4,2,14),

$$C(z) = \mathcal{E}(z) z^{-1} \mathcal{A}'(z) = z^{-1}(1-\mathcal{A})^{-1}\mathcal{A}' = z^{-1} \sum_{n \geq 0} \mathcal{A}^n \mathcal{A}', \tag{3.27}$$

and

$$G^n G' \overset{[,]}{=} \left(\frac{1}{n+1} G^{n+1} \right)', \quad \forall G \quad \blacksquare \tag{3.28}$$

Formula (3.16) can now be rewritten as

$$X_{L+1}(\ell n f) \approx 0. \tag{3.29}$$

Thus,

$$-\ell n f = -\ell n(1-\mathcal{A}(z)) = \sum_{n \geq 1} \frac{1}{n} \mathcal{A}(z)^n = \sum_{n=0}^{\infty} z^{n+1} H_n = \mathcal{H}(z) \tag{3.30}$$

is the generating series of integrals for our hierarchy.
Had we started with the mixed form (2.22),

$$X_2(A_n) = A'_{n+1} + \alpha A_n A'_0 + \beta A'_0 A_n, \quad \alpha + \beta = 1, \tag{3.31}$$

we would have:

$$X_2(f) = z^{-1} f' + \alpha f A'_0 + \beta A'_0 f \quad \Rightarrow \tag{3.32}$$

$$f^{-1} X_2(f) = z^{-1} f^{-1} f' + \alpha A'_0 + \beta f^{-1} A'_0 f \overset{[,]}{=} z^{-1} f^{-1} f' + A'_0, \tag{3.33}$$

since

$$f^{-1} A'_0 f \overset{[,]}{=} f f^{-1} A'_0 = A'_0.$$

Thus, the series (3.30) serves as the series of integrals also for the mixed system (3.31).
Remark 3.34.. By formulae (3.27,2,3),

$$zC(z) = -f^{-1} f'. \tag{3.35}$$

Therefore, by formulae (3.30,16):

$$\sum_{n=0}^{\infty} z^{n+1} X_{L+1}(H_n) = X_{L+1}(-\ell n f) \overset{[,]}{=} -z^{-L}(f^{-1}f' - \sum_{j=0}^{L-1} z^j z C_j) \overset{[,]}{=}$$

$$\overset{[,]}{=} z^{-L} \sum_{n \geq L} z^{n+1} H'_n = \sum_{n=0}^{\infty} z^{n+1} H'_{n+L} \quad \Rightarrow \tag{3.36}$$

$$X_{L+1}(H_n) \overset{[,]}{=} H'_{n+L}. \tag{3.37}$$

Thus, modulo commutators, the fluxes are themselves integrals.

Still more can be said about the integrals H_n's. Formula (2.23) implies that formula (2.17) is true for all the flows:

$$X_m(\mathcal{M}) = [D(\boldsymbol{X}_m), \mathcal{M}], \quad m \in \mathbf{Z}_{\geq 1}. \tag{3.38}$$

Therefore $\{Tr(\mathcal{M}^n), n \in \mathbf{Z}_{\geq 1}\}$ are integrals for all the flows.
Proposition 3.39.

$$H_n \overset{[,]}{=} \frac{1}{n+1} Tr(\mathcal{M}^{n+1}), \quad n \in \mathbf{Z}_{\geq 0}. \tag{3.40}$$

Proof. By formula (2.11a),

$$Tr(\mathcal{M}^L) = \sum_{s=0}^{L-1} \sum_{k=0}^{L-1-s} A_{L-1-k} \mathcal{E}_k = \sum_{k=0}^{L-1} (L-k) A_{L-1-k} \mathcal{E}_k = \sum_{r=0}^{L-1} (r+1) A_r \mathcal{E}_{L-1-r}. \tag{3.41}$$

Hence

$$\bar{\mathcal{H}}(z) = \sum_{n=0}^{\infty} z^{n+1} \frac{1}{n+1} Tr(\mathcal{M}^{n+1}) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \sum_{r=0}^{n} (r+1) A_r \mathcal{E}_{n-r} \quad \Rightarrow \tag{3.42}$$

$$\frac{\partial \bar{\mathcal{H}}}{\partial z} = \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} (r+1) A_r \mathcal{E}_{n-r} = \mathcal{A}_z \mathcal{E} = \mathcal{A}_z (1-\mathcal{A})^{-1}. \tag{3.43}$$

On the other hand, by formula (3.30),

$$\frac{\partial}{\partial z}(\sum_{n=0}^{\infty} z^{n+1} H_n) = \frac{\partial}{\partial z}(\sum_{n=1}^{\infty} \frac{1}{n} \mathcal{A}^n) \overset{[,]}{=} \sum_{n=1}^{\infty} (1-\mathcal{A})^{-1} \mathcal{A}_z, \tag{3.44}$$

and this is the same as (3.43) ■

Now, denote by $\delta H_n / \delta \boldsymbol{A}$ the column-vector of (noncommutative) variational derivatives of the Hamiltonian H_n:

$$\left(\frac{\delta H_n}{\delta \boldsymbol{A}} \right)_i = \frac{\delta H_n}{\delta A_i}, \quad n, i \in \mathbf{Z}_{\geq 0}. \tag{3.45}$$

The commutative theory suggests the formula

$$\frac{\delta H_{n+1}}{\delta \boldsymbol{A}} = \mathcal{M}^{\dagger}\Big(\frac{\delta H_n}{\delta \boldsymbol{A}}\Big), \tag{3.46}$$

where, by formula (2.6),

$$(\mathcal{M}^{\dagger})^j_i = (\mathcal{M}^i_j)^{\dagger} = (\delta^i_{j+1} + \delta^i_0 \hat{L}_{A_j})^{\dagger} = \delta^{j+1}_i + \delta^0_i \hat{R}_{A_j}. \tag{3.47}$$

Proposition 3.48.

$$\frac{\delta H_{n+1}}{\delta A_0} = \sum_j \frac{\delta H_n}{\delta A_j} A_j, \tag{3.49}$$

$$\frac{\delta H_{n+1}}{\delta A_{i+1}} = \frac{\delta H_n}{\delta A_i}. \tag{3.50}$$

Proof. Formulae (3.50) are equivalent to the relation

$$\frac{\delta H_n}{\delta A_i} = \frac{\delta H_{n-i}}{\delta A_0}, \tag{3.51}$$

and then formula (3.49) can be rewritten as

$$\frac{\delta H_{n+1}}{\delta A_0} = \sum_j \frac{\delta H_{n-j}}{\delta A_0} A_j, \tag{3.52}$$

By formula (3.30),

$$d(\mathcal{H}) = \sum_{n=1}^{\infty} \frac{1}{n} d(\mathcal{A}^n) = \sum_{n\geq 1} \frac{1}{n} \sum_{i+j=n-1} \mathcal{A}^i d(\mathcal{A}) \mathcal{A}^j \overset{[,]}{=} \sum_{n \geq 1} d(\mathcal{A})\mathcal{A}^{n-1} =$$

$$= d(\mathcal{A})(1-\mathcal{A})^{-1} = \sum_{k\geq 0} z^{k+1} dA_k (1-\mathcal{A})^{-1} \quad \Rightarrow \tag{3.53}$$

$$\frac{\delta \mathcal{H}}{\delta A_k} = z^{k+1}(1-\mathcal{A})^{-1} = z^k z (1-\mathcal{A})^{-1} = z^k \frac{\delta \mathcal{H}}{\delta A_0}, \tag{3.54}$$

which proves formula (3.51).

Next, formula (3.53) implies that

$$d(\mathcal{H})\mathcal{A} = \sum_{n\geq 1} \frac{1}{n} \sum_{i+j=n-1} \mathcal{A}^i d(\mathcal{A}) \mathcal{A}^j \mathcal{A} \overset{[,]}{=} d(\mathcal{A}) \sum_{n=1}^{\infty} \mathcal{A}^n =$$

$$= d(\mathcal{A})(\sum_{n=0}^{\infty} \mathcal{A}^n - 1) \overset{[,]}{=} d(\mathcal{H}) - d(\mathcal{A}) \quad \Rightarrow \tag{3.55}$$

$$\sum_{n\geq 0} z^{n+1} \frac{\delta H_n}{\delta A_0} \sum_{j \geq 0} z^{j+1} A_j = \sum_{m\geq 0} z^{m+1} \frac{\delta H_m}{\delta A_0} - z \quad \Rightarrow$$

$$\sum_{n+j=m} \frac{\delta H_n}{\delta A_0} A_j = \frac{\delta H_{n+1}}{\delta A_0}, \tag{3.56}$$

and this is formula (3.49) ■

The first few vectors $\delta H_n/\delta \boldsymbol{A}$ are:

$$\frac{\delta H_0}{\delta \boldsymbol{A}} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \tag{3.57a}$$

$$\frac{\delta H_1}{\delta \boldsymbol{A}} = \begin{pmatrix} A_0 \\ 1 \\ \mathbf{0} \end{pmatrix}, \tag{3.57b}$$

$$\frac{\delta H_2}{\delta \boldsymbol{A}} = \begin{pmatrix} A_0^2 + A_1 \\ A_0 \\ 1 \\ \mathbf{0} \end{pmatrix}, \tag{3.57c}$$

$$\frac{\delta H_3}{\delta \boldsymbol{A}} = \begin{pmatrix} A_0^3 + A_1 A_0 + A_0 A_1 + A_2 \\ A_0^2 + A_1 \\ A_0 \\ 1 \\ \mathbf{0} \end{pmatrix}. \tag{3.57d}$$

Formulae (2.12) and (3.52) show that

$$\frac{\delta H_n}{\delta A_0} = \mathcal{E}_n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{3.58}$$

4 Hydrodynamical Reductions

In this Section we verify that every flow in our hierarchy allows the two reductions

$$A_n = u^n h, \quad n \in \mathbf{Z}_{\geq 0}, \tag{4.1L}$$

$$A_n = h u^n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{4.1R}$$

We first dispose of the mixed flow

$$\dot{A}_n = A'_{n+1} + \alpha A_n A'_0 + \beta A'_0 A_n, \quad n \in \mathbf{Z}_{\geq 0}. \tag{4.2}$$

If

$$\dot{u} = u'u + \beta[h', u], \quad \dot{h} = (uh)' + \alpha h h' + \beta h' h, \tag{4.3}$$

then

$$\begin{aligned} (u^n h)^{\cdot} &= \sum_{i+j=n-1} u^i \dot{u} u^j h + u^n \dot{h} = \sum_{i+j=n-1} u^i (u'u + \beta h'u - \beta u h') u^j h + \\ &+ u^n [(uh)' + \alpha h h' + \beta h' h] = (u^n)' u h + \beta (h' u^n - u^n h') h + u^n (uh)' + \\ &+ \alpha u^n h h' + \beta u^n h' h = (u^{n+1} h)' + \alpha u^n h h' + \beta h' u^n h. \end{aligned} \tag{4.4}$$

Similarly, if

$$\dot{u} = uu' + \alpha[u, h'], \ \dot{h} = (hu)' + \alpha hh' + \beta h'h, \tag{4.5}$$

then

$$\begin{aligned}
&(hu^n)^{\cdot} = \dot{h}u^n + \sum_{i+j=n-1} hu^i \dot{u} u^j = [(hu)' + \alpha hh' + \beta h'h]u^n + \\
&+h \sum_{i+j=n-1} u^i [uu' + \alpha(uh' - h'u)]u^j = (hu)'u^n + \alpha hh'u^n + \beta h'hu^n + \\
&+hu(u^n)' + h\alpha(u^n h' - h'u^n) = (hu^{n+1})' + \beta h'hu^n + \alpha hu^n h'.
\end{aligned} \tag{4.6}$$

For the general flow X_L, we proceed by induction on L. Formula (2.9), in the form

$$\boldsymbol{X}_{L+1} = \mathcal{M}\boldsymbol{X}_L, \tag{4.7}$$

yields

$$\begin{aligned}
&X_{L+1}(A_i) = (\boldsymbol{X}_{L+1})_i = \sum_j \mathcal{M}_i^j (\boldsymbol{X}_L)_j = \sum_j (\delta_{i+1}^j + \delta_0^j A_i) X_L(A_j) = \\
&= X_L(A_{i+1}) + A_i X_L(A_0).
\end{aligned} \tag{4.8}$$

Therefore, for the case of the left representation,

$$\begin{aligned}
&X_{L+1}(A_{i+1} - uA_i) = X_{L+1}(A_{i+1}) - uX_{L+1}(A_i) - X_{L+1}(u)A_i = \\
&= X_L(A_{i+2}) + A_{i+1}X_L(A_0) - u\{X_L(A_{i+1}) + A_i X_L(A_0)\} - X_{L+1}(u)A_i = \\
&= X_L(A_{i+2} - uA_{i+1}) + X_L(u)A_{i+1} + (A_{i+1} - uA_i)X_L(A_0) - X_{L+1}(u)A_i = \\
&= X_L(A_{i+2} - uA_{i+1}) + (A_{i+1} - uA)X_L(A_0) + X_L(u)(A_{i+1} - uA_i) + \\
&+\{X_L(u)u - X_{L+1}(u)\}A_i.
\end{aligned} \tag{4.9}$$

Thus, if the flow X_L is left-reducible, then so is X_{L+1}, provided

$$X_{L+1}(u) = X_L(u)u. \tag{4.10}$$

But the flow X_1:

$$X_1(A_n) = A_n'. \tag{4.11}$$

is obviously left-reducible:

$$X_1(u) = u', \quad X_1(h) = h'. \tag{4.12}$$

The right-reduction case is almost as simple. We have:

$$\begin{aligned}
&X_{L+1}(A_{i+1} - A_i u) = X_{L+1}(A_{i+1}) - X_{L+1}(A_i)u - A_i X_{L+1}(u) = \\
&= X_L(A_{i+2}) + A_{i+1}X_L(A_0) - \{X_L(A_{i+1}) + A_i X_L(A_0)\}u - A_i X_{L+1}(u) = \\
&= X_L(A_{i+2} - A_{i+1}u) + A_{i+1}X_L(u) + (A_{i+1} - A_i u)X_L(A_0) + A_i[u, X_L(A_0)] - \\
&-A_i X_{L+1}(u),
\end{aligned} \tag{4.13}$$

and

$$A_{i+1}X_L(u) + A_i[u, X_L(A_0)] - A_i X_{L+1}(u) = (A_{i+1} - A_i u)X_L(u) + \\ +A_i\{uX_L(u) + [u, X_L(A_0)] - X_{L+1}(u)\}. \tag{4.14}$$

Thus, if the flow X_L is right-reducible, then so is X_{L+1}, provided

$$X_{L+1}(u) = uX_L(u) + [u, X_L(h)]. \tag{4.15}$$

The quantity $X_{L+1}(h)$ can be derived directly from formula (4.8) for $i = 0$. For the right-reduction case,

$$X_{L+1}(h) = X_L(hu) + hX_L(h) = hX_L(u) + (\hat{R}_u + \hat{L}_h)X_L(h), \tag{4.16}$$

and for the left-reduction case,

$$X_{L+1}(h) = X_L(uh) + hX_L(h) = \hat{R}_h X_L(u) + (u + h)X_L(h). \tag{4.17}$$

5 The Modified Hierarchies

What is the meaning of the hydrodynamical reductions (4.1)? And why do they exist? The complete answer is unknown. We may argue as follows.

For the case of the left reduction $\{A_n = u^n h\}$,

$$\mathcal{A}(z) = \sum z^{i+1} A_i = (1 - uz)^{-1} zh. \tag{5.1}$$

Thus, the series $\mathcal{A}(z)$ has a pole, of 1^{st} order. In the commutative case, we can impose the more general condition on $\mathcal{A}(z)$ of having a pole of order $N + 1$:

$$\mathcal{A}(z) = \sum_{k=0}^{N} (1 - uz)^{-k-1} z^{k+1} h_k. \tag{5.2}$$

As we shall see below, the noncommutativity forces N to be either 0 or ∞. The resulting left representation,

$$A_n = \sum_{k \geq 0} \binom{n}{k} u^k h_{n-k} \qquad n \in \mathbf{Z}_{\geq 0}, \tag{5.3}$$

amounts to the construction of the left-modified hierarchy.

Similar $N = 0$ or ∞ effect is shown below to operate for the case of the right reduction:

$$\mathcal{A}(z) = \sum_{k=0}^{N} z^{k+1} h_k (1 - uz)^{-k-1}, \tag{5.4}$$

$$A_n = \sum_{k \geq 0} \binom{n}{k} h_{n-k} u^k, \quad n \in \mathbf{Z}_{\geq 0}. \tag{5.5}$$

The corresponding right-modified hierarchy, as formula (4.15) indicates, is considerably more complicated than the left-modified one; its integrability is proved in §6.

We start with the left-reduced case first. Since the formulae (5.3) can be inverted:

$$h_n = \sum_{s=0}^{n} \binom{n}{s} (-1)^s u^s A_{n-s}, \tag{5.6}$$

the modified hierarchy is isomorphic to our original hierarchy supplemented with one extra field u governed by the evolution (4.10):

$$X_L(u) = u' u^{L-1}, \quad L \in \mathbf{Z}_{\geq 1}. \tag{5.7}$$

This is entirely similar to the Lax case $mKP \approx m + KP$ [9, Ch. 6].

Proposition 5.8.

$$X_2(h_n) = h'_{n+1} + h_n h'_0 + (n+1)u' h_n + u h'_n + \sum_{s=1}^{n-1} \binom{n}{s+1} [(-ad_u)^s (u')] h_{n-s}. \tag{5.9}$$

Corollary 5.10. The constrain

$$\{h_i = 0, \quad i \geq N+1\} \tag{5.10}$$

is possible only when $N = 0$: the last summand in formula (5.9) involves h_1 no matter what $n \geq 2$ is.

Proof of formula (5.9). Since $X_2(u) = u'u$, we find:

$$X_2(u^i) = (u^i)' u. \tag{5.11}$$

Therefore, the LHS of formula (5.9) is:

$$X_2(h_n) = X_2(\sum_{i=0}^{n} \binom{n}{i} (-1)^i u^i A_{n-i}) =$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \{(u^i)' u A_{n-i} + u^i A'_{n-i+1} + u^i A_{n-i} A'_0\}. \tag{5.12}$$

The RHS of formula (5.9), in the form

$$h'_{n+1} + h_n h'_0 + (u h_n)' + \sum_{s=0}^{n-1} \binom{n}{s+1} [(-ad_u)^s (u')] h_{n-s}, \tag{5.13}$$

is transformed into:

$$\sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j [(u^j)' A_{n+1-j} + u^j A'_{n+1-j}] + \tag{5.14}$$

$$+ \sum_{j=0}^{n} (-1)^j \binom{n}{j} [u^j A_{n-j} A'_0 + (u^{j+1})' A_{n-j} + u^{j+1} A'_{n-j}] + \tag{5.15}$$

$$+ \sum_{s \geq 0} \binom{n}{s+1} (-1)^s [(ad_u)^s (u')] \sum_{k=0}^{n-s} \binom{n-s}{k} (-1)^k u^k A_{n-s-k}. \tag{5.16}$$

The 2^{nd} sum in (5.14) plus the 3^{rd} one in (5.15), equal to the 2^{nd} one in (5.12).

The 3^{rd} sum in (5.12) equals to the 1^{st} one in (5.15).

The remaining sums: the 1^{st} ones in (5.12) and (5.14), the 2^{nd} one in (5.15), and the full (5.16), amount to the following identity in front of A_{n-r}:

$$(-1)^r \binom{n}{r} (u^r)' u \overset{?}{=} (-1)^{r+1} \binom{n+1}{r+1} (u^{r+1})' + (-1)^r \binom{n}{r} (u^{r+1})' + \\ + \sum_{s=0}^{r} (-1)^r \binom{n}{s+1} \binom{n-s}{r-s} [(ad_u))^s (u')] u^{r-s}. \tag{5.17}$$

Since

$$\binom{n}{s+1} \binom{n-s}{r-s} = \binom{n}{r} \binom{r}{s} \frac{n-s}{s+1}, \tag{5.18a}$$

$$\binom{n}{r+1} = \binom{n}{r} \frac{n-r}{r+1}, \tag{5.18b}$$

dividing (5.17) by $(-1)^r \binom{n}{r}$, we get:

$$(u^r)' u + \frac{n-r}{r+1} (u^{r+1})' \overset{?}{=} \sum_{s=0}^{r} \frac{n-s}{s+1} \binom{r}{s} [(ad_u)^s] (u')] u^{r-s}. \tag{5.19}$$

This is a polynomial in n, of degree =1, so the equality (5.19) is equivalent to a pair of identities

$$\frac{(u^{r+1})'}{r+1} \overset{?}{=} \sum_{s=0}^{r} \frac{1}{s+1} \binom{r}{s} [(ad_u)^s (u')] u^{r-s}, \tag{5.20}$$

$$(u^r)' u - (u^{r+1})' \overset{?}{=} - \sum_{s=0}^{r} \binom{r}{s} [(ad_u)^s (u')] u^{r-s}. \tag{5.21}$$

We handle the identity (5.20) first. It's linear in u', and $Ker(ad_u) = \{0\}$ on linear in u' polynomials; hence, (5.20) is equivalent to

$$ad_u \left(\frac{(u^{r+1})'}{(r+1)!} \right) \overset{?}{=} \sum_{s=0}^{r} \frac{(ad_u)^{s+1} (u')}{(s+1)!} \cdot \frac{u^{r-s}}{(r-s)!} = \\ = \sum_{s+i=r+1} \frac{(ad_u)^s (u')}{s!} \frac{u^i}{i!} - u' \frac{u^{r+1}}{(r+1)!}. \tag{5.22}$$

Formula (5.22) makes sense also for $r = -1$:

$$ad_u \left(\frac{(u^r)'}{r!} \right) \overset{?}{=} \sum_{s+i=r} \frac{(ad_u)^s (u')}{s} \frac{(u^i)}{i!} - u' \frac{u^r}{r!}. \tag{5.23}$$

Passing to the generating functions, we get:

$$ad_u((e^{uz})') \overset{?}{=} e^{z ad_u} (u') e^{uz} - u' e^{uz}. \tag{5.24}$$

But

$$ad_u((e^{uz})') = [e^{uz}, u']. \tag{5.25}$$

Indeed,

$$ad_u((e^{uz})') = ad_u(\sum_{i,j} \frac{z^{i+j+1}}{(i+j+1)!} u^i u' u^j) =$$

$$= \sum_{i,j} \frac{z^{i+j+1}}{(i+j+1)!}(u^{i+1}u'u^j - u^i u' u^{j+1}) = \sum_i \frac{z^{i+1}}{(i+1)!} u^{i+1}u' - \sum_j \frac{z^{j+1}}{(j+1)!} u' u^{j+1} =$$

$$= e^{uz}u' - u'e^{uz}. \tag{5.26}$$

Hence, formula (5.24) is equivalent to

$$e^{uz}u'e^{-uz} \stackrel{?}{=} e^{zad_u}(u'), \tag{5.27}$$

which is true, being the standard formula

$$Ad_{exp(uz)} = exp(zad_u). \tag{5.28}$$

The identity (5.21) is next. Rewrite it as

$$u^r u' \stackrel{?}{=} \sum_{s=0}^{r} \binom{r}{s} [(ad_u)^s(u')]u^{r-s} \Leftrightarrow \tag{5.29a}$$

$$\frac{u^r}{r!}u' \stackrel{?}{=} \sum \frac{(ad_u)^s(u')}{s!} \frac{u^{r-s}}{(r-s)!} \Leftrightarrow \tag{5.29b}$$

$$e^{uz}u' \stackrel{?}{=} e^{zad_u}(u')e^{uz} \Leftrightarrow \tag{5.29c}$$

$$e^{uz}u'e^{-uz} \stackrel{?}{=} e^{zad_u}(u'), \tag{5.29d}$$

and this is the same as (5.27) ■

We next prove the $N = 0$ or ∞ law for the right-modified case (5.5):

$$h_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_{n-i}u^i, \tag{5.30}$$

$$X_2(u) = uu' + [u, A_0'] = uu' + [u, h_0']. \tag{5.31}$$

Proposition 5.32.

$$X_2(h_n) = h_{n+1}' + h_n h_0' + (h_n u)' + \sum_{s\geq 0} \binom{n}{s+1} h_{n-s}(ad_u)^s(u'). \tag{5.33}$$

Proof. First, formula (5.31) implies that

$$X_2(u^i) = u(u^i)' + [u^i, A_0']. \tag{5.34}$$

Next, for the LHS of formula (5.33) we get:

$$X_2(h_n) = X_2(\sum_{i=0}^{n}(-1)^i\binom{n}{i}A_{n-i}u^i) =$$
$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\{A_{n-i}[u(u^i)' + u^iA_0' - A_0'u^i] + [A'_{n-i+1} + A_{n-i}A_0']u^i\}. \tag{5.35}$$

For the RHS of formula (5.33) we obtain:

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}[A'_{n+1-j}u^j + A_{n+1-j}(u^j)'] + \sum_{i=0}^{n}(-1)^i\binom{n}{i}A_{n-i}u^iA_0' + \tag{5.36}$$
$$+\sum_{i=0}^{n}(-1)^i\binom{n}{i}[A'_{n-i}u^{i+1} + A_{n-i}(u^{i+1})'] + \tag{5.37}$$
$$+\sum\binom{n}{s+1}\binom{n-s}{k}(-1)^kA_{n-s-k}u^k(ad_u)^s(u'). \tag{5.38}$$

The 3^{rd} and 5^{th} sums in (5.35) cancel out.

The 2^{nd} sum in (5.35) is the same as the last one in (5.36).

The 1^{st} sums in (5.36) and (5.37) combine to cancel out the 4^{th} sum in (5.35). The remaining terms, multiplying from the right A_{n-r}, become:

$$(-1)^r\binom{n}{r}u(u^r)' \overset{?}{=} (-1)^{r+1}\binom{n+1}{r+1}(u^{r+1})' + (-1)^r\binom{n}{r}(u^{r+1})' +$$
$$+\sum_s(-1)^r\binom{n}{s+1}\binom{n-s}{r-s}u^{r-s}(-ad_u)^s(u'). \tag{5.39}$$

But this is the mirror image of the identity (5.17), because the anti-involution $\mathcal{I}$ acting identically on the variables u and u', takes ad_u into $-ad_u$:

$$\mathcal{I}(ad_u(f)) = \mathcal{I}(uf - fu) = \mathcal{I}(f)u - u\mathcal{I}(f) = (-ad_u)\mathcal{I}(f) \quad \blacksquare \tag{5.40}$$

Remark 5.41. Comparing formula (5.13) and (5.33) we see that the left- and right-modified 2^{nd} flows are related by a mysterious anti-involution acting on the u's, but acting on the h_i's as an involution.

6 The Right-Modified Hierarchy

The right-modified hierarchy, in the variables

$$\{h_n, \quad n \in \mathbf{Z}_{\geq 0}\} \quad \& \quad u, \tag{6.1}$$

is isomorphic, through the map (5.30), to our original A-hierarchy extended by a single field u. This extended hierarchy will be determined once we construct an extension $\tilde{\mathcal{M}}$ of the matrix $\mathcal{M}$ (2.6). The desired extension can be extracted from formula (4.15),

$$X_{L+1}(u) = uX_L(u) + ad_uX_L(A_0), \tag{6.2}$$

which can be interpreted as the condition guaranteeing that the two-sided ideal generated by the expressions

$$A_{i+1} - A_i u, \quad i \in \mathbf{Z}_{\geq 0}, \tag{6.3}$$

is preserved by each flow in the hierarchy.

Thus, the extended matrix $\tilde{\mathcal{M}}$ is:

$$\tilde{\mathcal{M}} = \begin{array}{cc} & \begin{array}{ccc} u & A_0 & A_{j>0} \end{array} \\ \begin{array}{c} u \\ A_i \end{array} & \left(\begin{array}{c|cc} \widehat{L}_u & ad_u & 0 \\ \hline 0 & \multicolumn{2}{c}{\mathcal{M}} \end{array}\right) \end{array} \quad ; \tag{6.4}$$

We first verify that all the extended flows commute with the flow $\tilde{X}_2$:

$$\tilde{X}_2(u) = uu' + [u, A_0'], \tag{6.5a}$$

$$\tilde{X}_2(A_n) = X_2(A_n), \quad n \in \mathbf{Z}_{\geq 0}. \tag{6.5b}$$

We need to check formula (2.17):

$$\tilde{X}_2(\tilde{\mathcal{M}}) = [D(\tilde{\boldsymbol{X}}_2),\ \tilde{\mathcal{M}}]. \tag{6.6}$$

The LHS of formula (6.6) is

$$\tilde{X}_2(\tilde{\mathcal{M}}) = \begin{array}{cc} & \begin{array}{cc} u & A_j \end{array} \\ \begin{array}{c} u \\ A_i \end{array} & \begin{pmatrix} \widehat{L}_{uu'+[u,A_0']} & \delta_0^i ad_{uu'+[u,A_0']} \\ 0 & X_2(\mathcal{M})_i^j \end{pmatrix} \end{array}. \tag{6.7}$$

Next,

$$D(\tilde{\boldsymbol{X}}_2) = \begin{array}{cc} & \begin{array}{cc} u & A_j \end{array} \\ \begin{array}{c} u \\ A_i \end{array} & \begin{pmatrix} \widehat{R}_{u^0} + \widehat{L}_u \partial - ad'_{A_0} & \delta_0^i ad_u \partial \\ 0 & D(\boldsymbol{X}_2)_i^j \end{pmatrix} \end{array}, \tag{6.8}$$

so that

$$(D(\tilde{\boldsymbol{X}}_2)\tilde{\mathcal{M}})_u^u = (\hat{R}_{u'} + \hat{L}_u \partial - ad_{A_0'})\hat{L}_u, \tag{6.9a}$$

$$(D(\tilde{\boldsymbol{X}}_2)\tilde{\mathcal{M}})_u^k = \delta_0^k(\hat{R}_{u'} + \hat{L}_u \partial - ad_{A_0'})ad_u +$$

$$+ad_u\partial(\delta_1^k + \delta_0^k \hat{L}_{A_0}), \tag{6.9b}$$

$$(D(\tilde{\boldsymbol{X}}_2)\tilde{\mathcal{M}})_i^u = 0, \tag{6.9c}$$

$$(D(\tilde{\boldsymbol{X}}_2)\tilde{\mathcal{M}})_i^j = (D(\boldsymbol{X}_2)\mathcal{M})_i^j, \tag{6.9d}$$

$$(\tilde{\mathcal{M}}D(\tilde{\boldsymbol{X}}_2))_u^u = \hat{L}_u(\hat{R}_{u'} + \hat{L}_u\partial - ad_{A_0'}), \tag{6.10a}$$

$$(\tilde{\mathcal{M}}D(\tilde{\boldsymbol{X}}_2))_u^k = \delta_0^k \hat{L}_u ad_u \partial + ad_u[\delta_1^k \partial + \delta_0^k(\hat{R}_{A_0'} + \hat{L}_{A_0}\partial)], \tag{6.10b}$$

$$(\tilde{\mathcal{M}}D(\tilde{\boldsymbol{X}}_2))_i^u = 0, \tag{6.10c}$$

$$(\tilde{\mathcal{M}}D(\tilde{\boldsymbol{X}}_2))_i^j = (\mathcal{M}D(\boldsymbol{X}_2))_i^j. \tag{6.10d}$$

Hence, formula (6.6) is true at the A-rows. It is obviously true at the $u-u$ place:

$$[D(\tilde{\boldsymbol{X}}_2), \tilde{\mathcal{M}}]^u_u = [\delta^k_0(\hat{R}_{u'} + \hat{L}_u\partial - ad_{A'_0}), \hat{L}_u] = \hat{L}_u\hat{L}'_u - [\hat{L}_{A'_0}, \hat{L}_u] =$$
$$= \hat{L}_{uu'} + \hat{L}_{uA'_0 - A'_0 u} = \hat{L}_{uu'+[u,A'_0]} = \tilde{X}_2(\tilde{\mathcal{M}})^u_u. \tag{6.11}$$

At the $u - A_k$ place, we get:

$$[D(\tilde{\boldsymbol{X}}_2), \tilde{\mathcal{M}}]^k_u = \delta^k_0(\hat{R}_{u'} + \hat{L}_u\partial - ad_{A'_0})ad_u + ad_u\partial(\delta^k_1 + \delta^k_0\hat{L}_{A_0}) -$$
$$-\delta^k_0\hat{L}_u ad_u\partial - ad_u[\delta^k_1\partial + \delta^k_0(\hat{R}_{A'_0} + \hat{L}_{A_0}\partial)] = \delta^k_0 \ \text{ times :} \tag{6.12}$$

$$(\hat{R}_{u'} + \hat{L}_u\partial - ad_{A'_0})ad_u + ad_u\partial\hat{L}_{A_0} - \hat{L}_u ad_u\partial - ad_u(\hat{R}_{A'_0} + \hat{L}_{A_0}\partial). \tag{6.13}$$

The 2^{nd} and 5^{th} terms combine into

$$\hat{L}_u ad_{u'} = \hat{L}_u(\hat{L}_{u'} - \hat{R}_{u'}). \tag{6.14a}$$

The 4^{th} and 7^{th} terms yield

$$ad_u\hat{L}_{A'_0}, \tag{6.14b}$$

and together with the 6^{th} and 3^{rd} term make up

$$ad_u\hat{L}_{A'_0} - ad_u\hat{R}_{A'_0} - ad_{A'_0}ad_u = [ad_u, ad_{A'_0}] = ad_{[u,A'_0]}. \tag{6.14c}$$

The remaining 1^{st} terms in (6.13) and $(6.14a)$ combine into

$$\hat{R}_{u'}(\hat{L}_u - \hat{R}_u) + \hat{L}_u(\hat{L}_{u'} - \hat{R}_{u'}) = -\hat{R}_{u'}\hat{R}_u + \hat{L}_u\hat{L}_{u'} =$$
$$= -\hat{R}_{uu'} + \hat{L}_{uu'} = ad_{uu'}. \tag{6.14d}$$

Thus,

$$[D(\tilde{\boldsymbol{X}}_2), \tilde{\mathcal{M}}]^k_u = \delta^k_0 ad_{uu'+[u,A'_0]} = \tilde{X}_2(\tilde{\mathcal{M}}^k_u), \tag{6.15}$$

and formula (6.6) is thereby proven.

To show that all the extended flows commute, we verify formula (2.23), in the form

$$\widehat{\mathcal{O}\boldsymbol{V}}(\mathcal{O}\boldsymbol{U}) - \widehat{\mathcal{O}\boldsymbol{V}}(\mathcal{O}\boldsymbol{V}) = \mathcal{O}(\hat{V}(\mathcal{O})\boldsymbol{U} - \hat{U}(\mathcal{O})\boldsymbol{V}), \tag{6.16}$$

where $\mathcal{O}$ stands for $\tilde{\mathcal{M}}$ and

$$\boldsymbol{U} = \begin{pmatrix} f \\ \boldsymbol{Y} \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} g \\ \boldsymbol{Z} \end{pmatrix} \tag{6.17}$$

are arbitrary u- and A-independent vectors.

We have, by formula (6.4):

$$\mathcal{O}\boldsymbol{U} = \begin{pmatrix} uf + [u, Y_0] \\ \mathcal{M}\boldsymbol{Y} \end{pmatrix}, \quad \mathcal{O}\boldsymbol{V} = \begin{pmatrix} ug + [\mathcal{M}, Z_0] \\ \mathcal{M}\boldsymbol{Z} \end{pmatrix} \Rightarrow \tag{6.18}$$

$$\widehat{\mathcal{O}\boldsymbol{V}}(\mathcal{O}\boldsymbol{U}) = \widehat{\mathcal{O}\boldsymbol{V}}\begin{pmatrix} uf + [u, Y_0] \\ \mathcal{M}\boldsymbol{Y} \end{pmatrix} =$$
$$= \begin{pmatrix} (ug + [u, Z_0])f + [ug + [u, Z_0], Y_0] \\ \widehat{\mathcal{M}\boldsymbol{Z}}(\mathcal{M}\boldsymbol{Y}) \end{pmatrix}, \tag{6.19}$$

so that for the LHS of formula (6.16) we find:

$$\widehat{\mathcal{O}\boldsymbol{V}}(\mathcal{O}\boldsymbol{U}) - \widehat{\mathcal{O}\boldsymbol{U}}(\mathcal{O}\boldsymbol{V}) = \begin{pmatrix} F \\ \widehat{\mathcal{M}\boldsymbol{Z}}(\mathcal{M}\boldsymbol{Y}) - \widehat{\mathcal{M}\boldsymbol{Y}}(\mathcal{M}\boldsymbol{Z}) \end{pmatrix}, \tag{6.20}$$

where

$$F = (ug + [u, Z_0])f + [ug + [u, Z_0], Y_0] - (uf + [u, Y_0])g -$$
$$-[uf + [u, Y_0], Z_0] = u[g, f] - u[f, Z_0] + u[g, Y_0] +$$
$$+[u, [Z_0, Y_0]]. \tag{6.21}$$

Further, by formula (6.4),

$$\widehat{V}(\mathcal{O}) = \begin{array}{c c} & A_j \\ \begin{array}{c} u \\ A_i \end{array} & \left(\begin{array}{c|c} \widehat{L}_g & \delta_0^j ad_g \\ \hline 0 & \widehat{Z}(\mathcal{M}) \end{array}\right) \end{array} \Rightarrow \tag{6.22}$$

$$\hat{V}(\mathcal{O})\boldsymbol{U} = \hat{V}(\mathcal{O})\begin{pmatrix} f \\ \boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} gf + [g, Y_0] \\ \hat{Z}(\mathcal{M})\boldsymbol{Y} \end{pmatrix}, \tag{6.23}$$

whence for the RHS of formula (6.16) we get:

$$\mathcal{O}\begin{pmatrix} gf + [g, Y_0] - fg - [f, Z_0] \\ \hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\mathbf{Z} \end{pmatrix} = \begin{pmatrix} G \\ \mathcal{M}(\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z}) \end{pmatrix}, \tag{6.24}$$

where

$$G = u\{gf + [g, Y_0] - fg - [f, Z_0]\} + [u, (\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z})_0]. \tag{6.25}$$

But

$$(\hat{Z}(\mathcal{M})\boldsymbol{Y} - \hat{Y}(\mathcal{M})\boldsymbol{Z})_0 = \sum_j (\hat{Z}(\mathcal{M}_0^j)Y_j - \hat{Y}(\mathcal{M}_0^j)Z_j) =$$
$$= \sum_j (\delta_0^j \hat{L}_{Z_0} Y_j - \delta_0^j \hat{L}_{Y_0} Z_j) = [Z_0, Y_0], \tag{6.26}$$

because

$$\mathcal{M}_0^j = \delta_1^j + \delta_0^j \hat{L}_{A_0} \Rightarrow \hat{Z}(\mathcal{M}_0^j) = \delta_0^j \hat{L}_{Z_0}. \tag{6.27}$$

Substituting (6.26) into (6.25), we find:

$$G = u[g, f] + u[g, Y_0] - u[f, Z_0] + [u, [Z_0, Y_0]], \tag{6.28}$$

and this is the same expression as (6.21). Formula (6.16) is thereby proven.

7 A Single-Component Reduction

The 2^{nd} flow

$$X_2(A_k) = A'_{k+1} + A_k A'_0, \quad k \in \mathbf{Z}_{\geq 0}, \tag{7.1}$$

allows the reduction

$$A_k = (-1)^k A_0^{k+1}, \quad k \in \mathbf{Z}_{\geq 0}. \tag{7.2}$$

Indeed,

$$\dot{A}_0 = A'_1 + A_0 A'_0 = (-A_0^2)' + A_0 A'_0 = -A'_0 A_0, \tag{7.3}$$

and then

$$\begin{aligned} \dot{A}_k &= (-1)^k (A_0^{k+1})^{\cdot} = (-1)^k \sum_{i+j=k} A_0^i \dot{A}_0 A^j = (-1)^k \sum_{i+j=k} A_0^i (-A'_0 A_0) A_0^j = \\ &= (-1)^{k+1} (A_0^{k+1})' A_0 = (-1)^{k+1} [(A_0^{k+2})' - A_0^{k+1} A'_0] = A'_{k+1} + A_k A'_0. \end{aligned} \tag{7.4}$$

Let us show that the reduction (7.2) works for all the flows in our hierarchy. Since this reduction can be thought of as the composition of two reductions:

$$A_k = u^k h, \quad k \in \mathbf{Z}_{\geq 0}, \tag{7.5}$$

and

$$u = -h, \tag{7.6}$$

it is enough to show that two-component hierarchy resulting from the left reduction (7.5), itself allows the reduction (7.6).

By formulae (4.10,17),

$$\boldsymbol{X}_{L+1} = \begin{pmatrix} \hat{R}_u & 0 \\ \hat{R}_h & \hat{L}_{u+h} \end{pmatrix} \boldsymbol{X}_L, \quad L \in \mathbf{Z}_{\geq 0} : \tag{7.7}$$

$$X_{L+1}(u) = X_L(u) u, \tag{7.8a}$$

$$X_{L+1}(h) = X_L(u) h + (u+h) X_L(h). \tag{7.8b}$$

Hence,

$$X_{L+1}(u+h) = X_L(u)(u+h) + (u+h) X_L(h). \tag{7.9}$$

Thus, if the flow X_L is reducible then so is X_{L+1}; and X_1 is obviously reducible:

$$X_1(u+h) = (u+h)'. \tag{7.10}$$

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